Some Exact Solutions of Burgers Equation by Modified Homotopy Perturbation Method

Ghulam Mohiuddin[1] and Jamshad Ahmad[2]

Abstract

In this paper, Modified Homotopy Perturbation Method (MHPM) is applied to solve the generalized Burgers equation and exact solutions are found. MHPM has number of advantages over the classical methods and techniques as it requires simple iterations which are easily solvable and the solution obtained by this method converges rapidly to the exact or analytical solution. Some examples of Burgers Equations are also solved here to check the accuracy and efficiency of the method.

Keywords: Generalized Burgers Equation, Modified Homotopy Perturbation Method

1. Introduction

Mostly the phenomenon coming from the real world is in the form of the nonlinear partial differential equations. These equations are very difficult to solve when they appear in the form of system of equations as it requires a lot of complex calculations and are time consuming. Among these equations there is Burgers Equation which is a very fundamental equation in fluid mechanics. It is also present in many different applied mathematics branches as in acoustic waves, dynamics modeling, turbulence, shock wave formation and heat conduction [1-3]. Because of its applications in different applied mathematics fields, many mathematicians have developed different numerical methods and techniques for the numerical and exact solutions of Burgers equation such as Adomian Decomposition Method (ADM) [4-5], Variational Iteration Method (VIM) [6-7], Homotopy Analysis Method (HAM) [8-9], Reduced Differential Transform Method (RDTM) [10] and Homotopy Perturbation Method (HPM) [11].

In this paper, we have applied Modified Homotopy Perturbation Method (MHPM) for solving the Burgers (2+1), (3+1) and (n+1) Equations with given initial conditions and exact solutions are found. The MHPM has a major advantage other than the classical techniques and methods that solution is obtained after a very few iterations and this solution converges quickly to the exact and analytic solution. The reliability and accuracy of the proposed method is checked by solving some examples of Burgers Equation.

2. Basic Idea of Modified homotopy Perturbation Method (MHPM)

To understand the basic ideas of MHPM, consider the following nonlinear eqaution

$$Lu + Nu = 0, \qquad (1)$$

with the initial condition

$$u(x,0) = f(x).$$

Where L is the linear and N is the nonlinear operator. The variables of $u_0(x, t)$ can be separated as

$$u_0(x,t) = u(x,0)c_1(t) + \frac{\partial u(x,0)}{\partial x} c_2(t), \qquad (2)$$

[1] Department of Mathematics, NCBA&E, Gujrat (Campus), Pakistan, Email: mohiuddin.ghulam@yahoo.com

[2] Department of Mathematics, Faculty of Sciences, University of Gujrat, Pakistan

and the initial condition is given by

$$u(x,0)c_1(0)+\frac{\partial u(x,0)}{\partial x}c_2(0)=f(x). \qquad (3)$$

We obtain $c_1(t)$ and $c_2(t)$ by Eq. (2).

According to the Homotopy Perturbation Technique (HPM), a homotopy can be constructed as follows

$$H(v,p)=(1-p)(Lv-Lu_0)+p(Lv+Nv), \quad p\in[0,1], \qquad (4)$$

where $p \in [0, 1]$ is an embedding parameter and $u_0(x, t)$ is an initial approximation of Eq. (1) Now we have,

$$H(v,0)=Lv-Lu_0=0, \qquad H(v,1)=Lv+Nv=0.$$

The deformation of p from zero to unity is just that of v from u_0 to u, and $L(v - u_0)$ and $Lv + Nv$ are called homotopy. According to the HPM, we first use the embedding parameter p as a "small parameter", and assume that the solution to Eq. (4) may be expressed as a series in p

$$v=v_0+pv_1+p^2v_2+p^3v_3+\ldots \qquad (5)$$

Setting $p = 1$, the approximate solution to Eq. (4) is then

$$u=\lim_{p\to 1}v=v_0+v_1+v_2+v_3+\ldots$$

Substituting (5) into (4) and equating the terms with the same power of p, we get

$$\begin{aligned} &p^0, \quad Lv_0-Lu_0=0\\ &p^1, \quad Lv_1+Lu_0+Nv_0=0\\ &p^{k+1}, \quad Lv_{k+1}+Nv_k=0, \qquad k\geq 1. \end{aligned} \qquad (6)$$

Combining the initial approximation u_0 with the above equations, we identify v_n for n = 1, 2, $\cdots$, and obtain the n-th approximation of the exact solution as $u_n=\sum_{k=0}^{n}v_k$.

If there exists some $v_n=0,\ n\geq 1,$ then the exact solution to the equation can be denoted as $u(x,t)=\sum_{k=0}^{n-1}v_k$.

For simplicity, in this paper we assume that $v_1(x,t)=0,$ namely, the exact solution is denoted as $u(x,t)=v_0(x,t).$ since $u(x,t)$ satisfies the initial condition, we get

$$c_1(0)=1, \qquad c_2(0)=0.$$

Thus we have

$$Lv_1=-Lu_0-Nv_0=-L\left[u(x,0)c_1(t)+\frac{\partial u(x,0)}{\partial x}c_2(t)\right]-N\left[u(x,0)c_1(t)+\frac{\partial u(x,0)}{\partial x}c_2(t)\right]\equiv 0.$$

From the above formula, we get the proper c_1 (t) and c_2 (t). Furthermore, the appropriate initial approximation $u_0(x, t)$ may be obtained. The detailed process will be displayed in the next section.

3. Numerical Examples

In this section, we provide some examples of Burgers equation and find their exact solutions.

Example 3.1

Consider the following (2+1) dimensional Burgers Equation

$$u_t+\alpha(uu_x+uu_y)-\beta(u_{xx}+u_{yy})=0, \qquad (7)$$

with the initial condition

$$u(x,y,0)=x+y.$$

Let us choose

$$u_0(x,y,t)=(x+y)c_1(t)+\left[\frac{\partial u(x,y,0)}{\partial x}+\frac{\partial u(x,y,0)}{\partial y}\right]c_2(t)$$

$$u_0(x,y,t)=(x+y)c_1(t)+2c_2(t).$$

Then

$$\frac{\partial u_1}{\partial t}=-\frac{\partial u_0}{\partial t}-\alpha\left[u_0(u_0)_x+u_0(u_0)_y\right]+\beta\left[(u_0)_{xx}+(u_0)_{yy}\right]$$

$$\frac{\partial u_1}{\partial t}=-(x+y)c_1'(t)-2c_2'(t)-\alpha\left((x+y)c_1(t)+2c_2(t)\right)\left[2c_1(t)\right]+\beta(0)$$

$$\frac{\partial u_1}{\partial t}=-(x+y)c_1'(t)-2c_2'(t)-2\alpha(x+y)c_1^2(t)-2\alpha c_1(t)c_2(t)$$

$$=-(x+y)\left[c_1'(t)+2\alpha c_1^2(t)\right]-2(c_2'(t)+\alpha c_1(t)c_2(t))\equiv 0$$

From above equation we have a system of equations

$$c_1'(t)+2\alpha c_1^2(t)=0,\quad c_2'(t)+\alpha c_1(t)c_2(t),\quad c_2(t)=0,\quad c_1(0)=1,\quad c_2(0)=0.$$

Solving this equation by HPM, we get

$$c_1(t)=\frac{1}{1+2\alpha t}.$$

So the solution of the Eq. (7) is

$$u(x,y,t)=u_0(x,y,t)$$

$$=\frac{x+y}{1+2\alpha t}.$$

This is the exact solution of the Eq. (7).

Example 3.2

Next consider the (3+1) dimensional Burgers Equation

$$u_t+\alpha(uu_x+uu_y+uu_z)-\beta(u_{xx}+u_{yy}+u_{zz})=0, \quad (8)$$

with the initial condition

$$u(x,y,z,0)=x+y+z.$$

Let us choose

$$u_0(x,y,z,t)=(x+y+z)c_1(t)+\left[\frac{\partial u(x,y,z,0)}{\partial x}+\frac{\partial u(x,y,z,0)}{\partial y}+\frac{\partial u(x,y,z,0)}{\partial z}\right]c_2(t)$$

$$=(x+y+z)c_1(t)+3c_2(t).$$

Then

$$(u_1)_t=-(u_0)_t-\alpha\left[u_0(u_0)_x+u_0(u_0)_y+u_0(u_0)_z\right]+\beta\left[(u_0)_{xx}+(u_0)_{yy}+(u_0)_{zz}\right]$$

$$=-(x+y+z)c_1'(t)-3c_2'(t)-\alpha\left[(x+y+z)c_1(t)+3c_2(t)\right]\left[3c_2(t)\right]+\beta(0)$$

$$=-(x+y+z)c_1'(t)-3c_2'(t)-3\alpha(x+y+z)c_1^2(t)-9\alpha c_1(t)c_2(t)\equiv 0$$

$$=-(x+y+z)\left[c_1'(t)+3\alpha c_1^2(t)\right]-3\left[c_2'(t)+3\alpha c_1(t)c_2(t)\right]\equiv 0.$$

From above equation we have a system of equations

$$c_1'(t)+3\alpha c_1^2(t)=0,\quad c_2'(t)+3\alpha c_1(t)c_2(t),\quad c_1(0)=1,\quad c_2(t)=0,\quad c_2(0)=0$$

Solving the above nonlinear equation by HPM we get the solution

$$c_1(t)=\frac{1}{1+3\alpha t}.$$

So the solution of the Eq. (8) is

$$u(x,y,z,t) = u_0(x,y,z,t)$$
$$= \frac{x+y+z}{1+3\alpha t}.$$

This is the exact solution of the Eq. (8).

Example 3.3

In the last, consider the following (n+1) dimensional Burgers Equation

$$u_t + \alpha\left[u\frac{\partial u}{\partial x_1} + u\frac{\partial u}{\partial x_2} + u\frac{\partial u}{\partial x_3} + \ldots + u\frac{\partial u}{\partial x_n}\right] - \beta\left[\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2}\right], \qquad (9)$$

with the initial condition

$$u(x_1,x_2,x_3,\ldots x_n,t) = x_1 + x_2 + x_3 + \ldots x_n.$$

Let us choose

$$u_0(x_1,x_2,x_3,\ldots x_n,t) = (x_1+x_2+x_3+\ldots x_n)c_1(t) + \left[\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3} + \ldots \frac{\partial u}{\partial x_n}\right]c_2(t)$$
$$= (x_1+x_2+x_3+\ldots x_n)c_1(t) + nc_2(t).$$

Then we have

$$\frac{\partial u_1}{\partial t} = -\frac{\partial u_0}{\partial t} - \alpha\left[u_0\frac{\partial u_0}{\partial x_1} + u_0\frac{\partial u_0}{\partial x_2} + u_0\frac{\partial u_0}{\partial x_3} + \ldots u_0\frac{\partial u_0}{\partial x_n}\right] + \beta\left[\frac{\partial^2 u_0}{\partial x_1^2} + \frac{\partial^2 u_0}{\partial x_2^2} + \ldots \frac{\partial^2 u_0}{\partial x_n^2}\right]$$
$$= -(x_1+x_2+x_3+\ldots x_n)c_1'(t) - nc_2'(t) - \alpha\left[(x_1+x_2+x_3+\ldots x_n)c_1(t) + nc_2(t)\right]\left[nc_1(t)\right] + \beta(0)$$
$$= -(x_1+x_2+x_3+\ldots x_n)\left[c_1'(t) + \alpha nc_1^2(t)\right] - n\left[c_2'(t) + nc_1(t)c_2(t)\right].$$

From above equation we have a system of equations

$$c_1'(t) + \alpha nc_1^2(t) = 0 \qquad (10)$$
$$c_2'(t) + nc_1(t)c_2(t) = 0, \qquad c_1(0) = 1, \qquad c_2(t) = 0, \qquad c_2(0) = 0.$$

Solving the nonlinear Eq. (10) by HPM, we get the solution of (10)

$$c_1(t) = \frac{1}{1+n\alpha t}.$$

So the solution of the Eq. (9) is

$$u(x_1,x_2,x_3,\ldots x_n,t) = u_0(x_1,x_2,x_3,\ldots x_n,t)$$
$$= \frac{x_1+x_2+x_3+\ldots+x_n}{1+\alpha nt}.$$

This is the exact solution of the Eq. (9).

4. Conclusion

In this present work, we have applied MHPM to find the exact solutions of the Burgers equation.

The results gained through this method are quite efficient and reliable. Applying MHPM proves to a powerful scheme for finding the analytic and exact solutions of the nonlinear PDEs and shows the satisfactory results.

References

- M. Basto, V. Semiao, F. Calherios, Dynamics and synchronization of numerical solutions of the Burgers equation, J. Comput. Appl. Math. 23, 793-806 (2009)

- A. M. Wazwaz, Analytic study on Burgers, Fisher, Huxley equations and combined forms of these equations, Appl. Math. Comp., 195(2), pp.754-761 (2008)
- M. M. Rashidi, E. Erfani, New analytical method for solving Burgers' and nonlinear heat transfer equations and comparison with HAM, Comput. Phys. Commun. 180, 1539–1544, (2009)
- S. Abbasbandy, M. T. Darvishi, A numerical solution of Burgers' equation by time discretization of Adomian's decomposition method, Appl. Math. Comput. 170, 95–102 (2005)
- D. Kaya and A. Yokus, A numerical comparison of partial solutions in the decomposition method for linear and nonlinear partial differential equations, Math. Comput. Simul. 60, 507–512 (2002)
- J. Biazar, H. Aminikhah, Exact and numerical solutions for non-linear Burger's equation by VIM, Math. Comput. Modelling 49, 1394–1400 (2009)
- A. M. Wazwaz, The variational iteration method for rational solutions for KdV, K(2,2), Burgers, and cubic Boussinesq equations, Journal of Computational Applied Mathematics (in press)
- S. Abbasbandy, The application of homotopy analysis method to solve a generalized Hirota-Satsuma coupled KdV equation, Physics Letters A 361, 478 – 483 (2007)
- A. K. Alomari, M. S. M. Noorani and R. Nazar, The Homotopy Analysis Method for the Exact Solutions of the K(2,2), Burgers and Coupled Burgers Equations, Appl. Math. Sci., Vol. 2, no. 40, 1963 – 1977 (2008)
- A. Kumar, R. Arora, Solutions of the coupled system of Burgers' equations and coupled Klein-Gordon equation by RDT Method, Int. J. Adv. Appl. Math. & Mech. Volume 1, Issue 2, pp. 133-145 (2013)
- J. H. He, Homotopy perturbation method, A new nonlinear technique, Appl. Math. and Compt. 135, pp. 73–79 (2003)